2. E; $a_n = \frac{2}{(n+1)(n+2)} = \frac{2}{n+1} - \frac{2}{n+2}$

$$S_1 = \left(\frac{2}{2} - \frac{2}{3}\right) = 1 - \frac{2}{3}$$
$$S_2 = \left(1 - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{2}{4}\right) = 1 - \frac{2}{4}$$
$$\vdots$$
$$S_n = \left(1 - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{2}{4}\right) + \cdots + \left(\frac{2}{n+1} - \frac{2}{n+2}\right)$$
$$= 1 - \frac{2}{n+2}$$
$$S = \lim_{n\to\infty}\left(1 - \frac{2}{n+2}\right)$$
$$= 1$$

3. D; $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges by the Integral Test since $\int_1^{\infty} \frac{\ln x}{x}\,dx = \lim_{k\to\infty} [(\ln x)^2]_1^k = \infty$, so $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}$ does not converge absolutely. But it converges by the Alternating Series Test: $\left(\text{Use } \frac{d}{dx}\left(\frac{\ln x}{x}\right) = \frac{1-\ln x}{x^2} \text{ to show the}\right.$ $\left. u_n \text{ are decreasing for } n \ge 3.\right)$

4. **(a)** Ratio test:

$$\lim_{n\to\infty} \frac{(n+1)|2x+3|^{n+1}}{n+3} \cdot \frac{n+2}{n|2x+3|^n}$$
$$= \lim_{n\to\infty} \frac{(n+1)(n+2)}{(n+3)(n)} \cdot |2x+3|$$
$$= |2x+3|$$
$$|2x+3| < 1 \Rightarrow -2 < x < -1$$

The series converges absolutely on $(-2, -1)$.

(b) The series diverges at both endpoints by the nth-Term Test:

$$\lim_{n\to\infty} \frac{n(2(-2)+3)^n}{n+2} \ne 0 \text{ and}$$
$$\lim_{n\to\infty} \frac{n(2(-1)+3)^n}{n+2} \ne 0.$$

Since the series converges absolutely on $(-2, -1)$ and diverges at both endpoints, there are no values of x for which the series converges conditionally.

Chapter 10 Review Exercises (pp. 531–534)

1. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|-x|^{n+1}}{(n+1)!} \cdot \frac{n!}{|-x|^n}$

$$= \lim_{n\to\infty} \frac{|x|}{n+1}$$
$$= 0$$

The series converges absolutely for all x.

(a) ∞

(b) All real numbers

(c) All real numbers

(d) None

2. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x+4|^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{|x+4|^n}$

$$= \frac{|x+4|}{3}.$$

The series converges absolutely for $\frac{|x+4|}{3} < 1$, or $-7 < x < -1$.

Check $x = -7$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

Check $x = -1$: diverges.

(a) 3

(b) $[-7, -1)$

(c) $(-7, -1)$

(d) At $x = -7$

3. This is a geometric series, so it converges absolutely when $|r| < 1$ and diverges for all other values of x. Since $r = \frac{2}{3}(x-1)$, the series converges absolutely when $\left|\frac{2}{3}(x-1)\right| < 1$, or $-\frac{1}{2} < x < \frac{5}{2}$.

(a) $\frac{3}{2}$

(b) $\left(-\frac{1}{2}, \frac{5}{2}\right)$

(c) $\left(-\frac{1}{2}, \frac{5}{2}\right)$

(d) None

4. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x-1|^{2n}}{(2n+1)!}\cdot\frac{(2n-1)!}{|x-1|^{2n-2}}$

$= \lim_{n\to\infty}\frac{|x-1|^2}{(2n+1)(2n)}$

$= 0$

The series converges absolutely for all x.

(a) ∞

(b) All real numbers

(c) All real numbers

(d) None

5. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|3x-1|^{n+1}}{(n+1)^2}\cdot\frac{n^2}{|3x-1|^n}$

$= |3x-1|$

The series converges absolutely for $|3x-1| < 1,$ or $0 < x < \frac{2}{3}.$ Furthermore, when $|3x-1| = 1,$ we have $|a_n| = \frac{1}{n^2}$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges by the p-Test with $p = 2,$ so $\sum_{n=1}^{\infty}a_n$ also converges absolutely at the endpoints.

(a) $\frac{1}{3}$

(b) $\left[0, \frac{2}{3}\right]$

(c) $\left[0, \frac{2}{3}\right]$

(d) None

6. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+2)|x|^{3n+3}}{(n+1)|x|^{3n}} = |x|^3.$ The series converges absolutely for $|x|^3 < 1,$ or $-1 < x < 1$. The series diverges for $|x| > 1$. When $|x| = 1,$ the series diverges by the nth-Term Test.

(a) 1

(b) $(-1, 1)$

(c) $(-1, 1)$

(d) None

7. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$

$= \lim_{n\to\infty}\frac{(n+2)|2x+1|^{n+1}}{(2n+3)2^{n+1}}\cdot\frac{(2n+1)2^n}{(n+1)|2x+1|^n}$

$= \frac{|2x+1|}{2}$

The series converges absolutely for $\frac{|2x+1|}{2} < 1,$ or $-\frac{3}{2} < x < \frac{1}{2};$ the series diverges for $\frac{|2x+1|}{2} > 1.$ When $\frac{|2x+1|}{2} = 1,$ the series diverges by the nth-Term Test.

(a) 1

(b) $\left(-\frac{3}{2}, \frac{1}{2}\right)$

(c) $\left(-\frac{3}{2}, \frac{1}{2}\right)$

(d) None

8. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x|^{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{|x|^n}$

$= |x|\lim_{n\to\infty}\frac{n^n}{(n+1)(n+1)^n}$

$= |x|\lim_{n\to\infty}\frac{1}{(n+1)\left(1+\frac{1}{n}\right)^n}$

$= \frac{|x|}{e}\lim_{n\to\infty}\frac{1}{n+1}$

$= 0$

The series converges absolutely for all x. Another way to see that the series must converge is to observe that for $n \ge 2x,$ we have $\left|\frac{x^n}{n^n}\right| \le \left(\frac{1}{2}\right)^n,$ so the terms are (eventually) bounded by the terms of a convergent geometric series.A third way to solve this exercise is to use the nth-Root Test (see Exercises 73–74 in Section 10.5).

(a) ∞

(b) All real numbers

(c) All real numbers

(d) None

9. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x|^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{|x|^n} = |x|$

The series converges absolutely for $|x|<1$, or $-1<x<1$.
Check $x=-1$:
$\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ converges by the Alternating Series Test.
Check $x=1$:
$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges by the p-Test with $p=\frac{1}{2}$.

(a) 1

(b) $[-1, 1)$

(c) $(-1, 1)$

(d) At $x=-1$

10. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{e^{n+1}|x|^{n+1}}{(n+1)^e}\cdot\frac{n^e}{e^n|x|^n} = e|x|.$

The series converges absolutely for $e|x|<1$, or $-\frac{1}{e}<x<\frac{1}{e}$. Furthermore, when $e|x|=1$, we have $|a_n|=\frac{1}{n^e}$ and $\sum_{n=1}^{\infty}\frac{1}{n^e}$ converges by the p-Test with $p=e$, so $\sum_{n=1}^{\infty}a_n$ also converges absolutely at the interval endpoints.

(a) $\frac{1}{e}$

(b) $\left[-\frac{1}{e}, \frac{1}{e}\right]$

(c) $\left[-\frac{1}{e}, \frac{1}{e}\right]$

(d) None

11. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$

$= \lim_{n\to\infty}\frac{(n+2)|x|^{2n+1}}{3^{n+1}}\cdot\frac{3^n}{(n+1)|x|^{2n-1}}$

$= \frac{x^2}{3}.$

The series converges absolutely when $\frac{x^2}{3}<1$, or $-\sqrt{3}<x<\sqrt{3}$; the series diverges when $x^2>3$.
When $|x|=\sqrt{3}$, the series diverges by the nth-Term Test.

(a) $\sqrt{3}$

(b) $\left(-\sqrt{3}, \sqrt{3}\right)$

(c) $\left(-\sqrt{3}, \sqrt{3}\right)$

(d) None

12. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x-1|^{2n+3}}{2n+3}\cdot\frac{2n+1}{|x-1|^{2n+1}}$

$= |x-1|^2.$

The series converges absolutely when $|x-1|^2<1$, or $0<x<2$.

Check $x=0$: $\sum_{n=0}^{\infty}\frac{(-1)^n(-1)^{2n+1}}{2n+1} = -\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$ converges conditionally by the Alternating Series Test.

Check $x=2$: $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$ converges conditionally by the Alternating Series Test.

(a) 1

(b) $[0, 2]$

(c) $(0, 2)$

(d) At $x=0$ and $x=2$

13. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)!|x|^{2n+2}}{2^{n+1}}\cdot\frac{2^n}{n!|x|^{2n}}$

$= \lim_{n\to\infty}\frac{(n+1)x^2}{2}$

$= \begin{cases} 0, & x = 0 \\ \infty, & x \neq 0 \end{cases}$

The series converges only at $x = 0$.

(a) 0

(b) $x = 0$ only

(c) $x = 0$

(d) None

14. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|10x|^{n+1}}{\ln(n+1)}\cdot\frac{\ln n}{|10x|^n} = |10x|$

The series converges absolutely for $|10x| < 1$, or $-\frac{1}{10} < x < \frac{1}{10}$.

Check $x = -\frac{1}{10}$: $\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln n}$ converges by the Alternating Series Test.

Check $x = \frac{1}{10}$: $\sum_{n=2}^{\infty}\frac{1}{\ln n}$ diverges by the Direct Comparison Test, since $\frac{1}{\ln n} > \frac{1}{n}$ for $n \geq 2$ and $\sum_{n=2}^{\infty}\frac{1}{n}$ diverges.

(a) $\frac{1}{10}$

(b) $\left[-\frac{1}{10}, \frac{1}{10}\right)$

(c) $\left(-\frac{1}{10}, \frac{1}{10}\right)$

(d) At $x = -\frac{1}{10}$

15. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+2)!|x|^{n+1}}{(n+1)!|x|^n}$

$= \lim_{n\to\infty}(n+2)|x|$

$= \infty \ (x \neq 0)$

The series converges only at $x = 0$.

(a) 0

(b) $x = 0$ only

(c) $x = 0$

(d) None

16. This is geometric series with $r = \frac{x^2 - 1}{2}$, so it converges absolutely when $\left|\frac{x^2-1}{2}\right| < 1$, or $-\sqrt{3} < x < \sqrt{3}$. It diverges for all other values of x.

(a) $\sqrt{3}$

(b) $\left(-\sqrt{3}, \sqrt{3}\right)$

(c) $\left(-\sqrt{3}, \sqrt{3}\right)$

(d) None

17. $f(x) = \frac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^n x^n + \cdots$, evaluated at $x = \frac{1}{4}$. Sum $= \frac{1}{1+\left(\frac{1}{4}\right)} = \frac{4}{5}$.

18. $f(x) = \ln(1+x)$

$= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n}$,

evaluated at $x = \frac{2}{3}$.

Sum $= \ln\left(1 + \frac{2}{3}\right) = \ln\left(\frac{5}{3}\right)$.

19. $f(x) = \sin x$

$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \cdots$,

evaluated at $x = \pi$, Sum $= \sin \pi = 0$.

20. $f(x) = \cos x$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots,$$

evaluated at $x = \frac{\pi}{3}$. Sum $= \cos\frac{\pi}{3} = \frac{1}{2}$.

21. $f(x) = e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots,$

evaluated at $x = \ln 2$. Sum $= e^{\ln 2} = 2$.

22. $f(x) = \tan^{-1} x$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots,$$

evaluated at $x = \frac{1}{\sqrt{3}}$. Sum $= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$. (Note that when n is replaced by $n-1$, the general term of $\tan^{-1} x$ becomes $(-1)^{n-1}\frac{x^{2n-1}}{2n-1}$, which matches the general term given in the exercise.)

23. Replace x by $6x$ in the Maclaurin series for $\frac{1}{1-x}$ given at the end of Section 10.2.

$$\frac{1}{1-6x} = 1 + (6x) + (6x)^2 + \cdots + (6x)^n + \cdots$$
$$= 1 + 6x + 36x^2 + \cdots + (6x)^n + \cdots$$

24. Replace x by x^3 in the Maclaurin series for $\frac{1}{1+x}$ given at the end of Section 10.2.

$$\frac{1}{1+x^3} = 1 - (x^3) + (x^3)^2 - \cdots + (-x^3)^n + \cdots$$
$$= 1 - x^3 + x^6 - \cdots + (-1)^n x^{3n} + \cdots$$

25. The Maclaurin series for a polynomial is the polynomial itself: $1 - 2x^2 + x^9$.

26. $\frac{4x}{1-x} = 4x\left(\frac{1}{1-x}\right)$

$$= 4x(1 + x + x^2 + \cdots + x^n + \cdots)$$
$$= 4x + 4x^2 + 4x^3 + \cdots + 4x^{n+1} + \cdots$$

27. Replace x by πx in the Maclaurin series for $\sin x$ given at the end of Section 10.2.

$$\sin \pi x = \pi x - \frac{(\pi x)^3}{3!} + \frac{(\pi x)^5}{5!} - \cdots + (-1)^n \frac{(\pi x)^{2n+1}}{(2n+1)!} + \cdots$$

28. Replace x by $\frac{2x}{3}$ in the Maclaurin series for $\sin x$ given at the end of Section 10.2.

$$-\sin\frac{2x}{3} = -\left(\frac{2x}{3} - \frac{\left(\frac{2x}{3}\right)^3}{3!} + \frac{\left(\frac{2x}{3}\right)^5}{5!} - \cdots + (-1)^n \frac{\left(\frac{2x}{3}\right)^{2n+1}}{(2n+1)!}\right)$$

$$= -\frac{2x}{3} + \frac{4x^3}{81} - \frac{4x^5}{3645} + \cdots + \frac{(-1)^{n+1}\left(\frac{2x}{3}\right)^{2n+1}}{(2n+1)!}$$

29. $$-x + \sin x = -x + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots\right)$$

$$= -\frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$$

30. $$\frac{e^x + e^{-x}}{2} = \frac{1}{2}\left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots\right) + \frac{1}{2}\left(1 - x + \frac{x^2}{2!} + \cdots + (-1)^n \frac{x^n}{n!} + \cdots\right)$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots$$

31. Replace x by $\sqrt{5x}$ in the Maclaurin series for $\cos x$ given at the end of Section 10.2.

$$\cos\sqrt{5x} = 1 - \frac{\left(\sqrt{5x}\right)^2}{2!} + \frac{\left(\sqrt{5x}\right)^4}{4!} - \cdots + (-1)^n \frac{\left(\sqrt{5x}\right)^{2n}}{(2n)!} + \cdots$$

$$= 1 - \frac{5x}{2!} + \frac{(5x)^2}{4!} - \cdots + (-1)^n \frac{(5x)^n}{(2n)!} + \cdots$$

32. Replace x by $\frac{\pi x}{2}$ in the Maclaurin series for e^x given at the end of Section 10.2.

$$e^{\pi x/2} = 1 + \frac{\pi x}{2} + \frac{\left(\frac{\pi x}{2}\right)^2}{2!} + \cdots + \frac{\left(\frac{\pi x}{2}\right)^n}{n!} + \cdots$$

$$= 1 + \frac{\pi x}{2} + \frac{\pi^2 x^2}{8} + \cdots + \frac{1}{n!}\left(\frac{\pi x}{2}\right)^n + \cdots$$

33. Replace x by $-x^2$ in the Maclaurin series for e^x given at the end of Section 10.2, and multiply the resulting series by x.

$$xe^{-x^2} = x\left[1 + (-x^2) + \frac{(-x^2)^2}{2!} + \cdots + \frac{(-x^2)^n}{n!} + \cdots\right]$$

$$= x - x^3 + \frac{x^5}{2!} - \cdots + (-1)^n \frac{x^{2n+1}}{n!} + \cdots$$

34. Replace x by $3x$ in the Maclaurin series for $\tan^{-1} x$ given at the end of Section 10.2.

$$\tan^{-1} 3x = 3x - \frac{(3x)^3}{3} + \frac{(3x)^5}{5} - \cdots + (-1)^n \frac{(3x)^{2n+1}}{2n+1} + \cdots$$

35. Replace x by $-2x$ in the Maclaurin series for $\ln(1+x)$ given at the end of Section 10.2.

$$\ln(1-2x) = -2x - \frac{(-2x)^2}{2} + \frac{(-2x)^3}{3} - \cdots + (-1)^{n-1}\frac{(-2x)^n}{n} + \cdots$$
$$= -2x - 2x^2 - \frac{8x^3}{3} - \cdots - \frac{(2x)^n}{n} - \cdots.$$

36. Use the Maclaurin series for $\ln(1+x)$ given at the end of Section 10.2.

$$x\ln(1-x) = x\ln\left[1+(-x)\right]$$
$$= x\left[-x - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \cdots + (-1)^{n-1}\frac{(-x)^n}{n} + \cdots\right]$$
$$= -x^2 - \frac{x^3}{2} - \frac{x^4}{3} - \cdots - \frac{x^{n+1}}{n} - \cdots$$

37. $f(2) = (3-x)^{-1}\Big|_{x=2} = 1$

$f'(2) = (3-x)^{-2}\Big|_{x=2} = 1$

$f''(2) = 2(3-x)^{-3}\Big|_{x=2} = 2$, so $\dfrac{f''(2)}{2!} = 1$

$f'''(2) = 6(3-x)^{-4}\Big|_{x=2} = 6$, so $\dfrac{f'''(2)}{3!} = 1$

$f^{(n)}(2) = n!(3-x)^{-n-1}\Big|_{x=2} = n!$, so $\dfrac{f^{(n)}(2)}{n!} = 1$

$$\frac{1}{3-x} = 1 + (x-2) + (x-2)^2 + (x-2)^3 + \cdots + (x-2)^n + \cdots$$

38. $f(-1) = (x^3 - 2x^2 + 5)\Big|_{x=-1} = 2$

$f'(-1) = (3x^2 - 4x)\Big|_{x=-1} = 7$

$f''(-1) = (6x-4)\big|_{x=-1} = -10$, so $\dfrac{f''(-1)}{2!} = -5$

$f'''(-1) = 6\big|_{x=-1} = 6$, so $\dfrac{f'''(-1)}{3!} = 1$

$f^{(n)}(-1) = 0$ for $n \geq 4$.

$x^3 - 2x^2 + 5 = 2 + 7(x+1) - 5(x+1)^2 + (x+1)^3$

This is a finite series and the general term for $n \geq 4$ is 0.

39. $f(3) = \frac{1}{x}\Big|_{x=3} = \frac{1}{3}$

$f'(3) = -x^{-2}\big|_{x=3} = -\frac{1}{9}$

$f''(3) = 2x^{-3}\big|_{x=3} = \frac{2}{27}$, so $\frac{f''(3)}{2!} = \frac{1}{27}$

$f'''(3) = -6x^{-4}\big|_{x=3} = -\frac{2}{27}$, so $\frac{f'''(3)}{3!} = -\frac{1}{81}$

$f^{n}(3) = (-1)^n n!\, x^{-n-1}\Big|_{x=3} = \frac{(-1)^n n!}{3^{n+1}}$, so

$\frac{f^{(n)}(3)}{n!} = \frac{(-1)^n}{3^{n+1}}$

$$\frac{1}{x} = \frac{1}{3} - \frac{1}{9}(x-3) + \frac{1}{27}(x-3)^2 - \frac{1}{81}(x-3)^3 + \cdots + (-1)^n \frac{(x-3)^n}{3^{n+1}} + \cdots$$

40. $f(\pi) = \sin x\big|_{x=\pi} = 0$

$f'(\pi) = \cos x\big|_{x=\pi} = -1$

$f''(\pi) = -\sin x\big|_{x=\pi} = 0$, so $\frac{f''(\pi)}{2!} = 0$

$f'''(\pi) = -\cos x\big|_{x=\pi} = 1$, so $\frac{f'''(\pi)}{3!} = \frac{1}{6}$

$$f^{(k)}(\pi) = \begin{cases} 0, & \text{if } k \text{ is even} \\ -1, & \text{if } k = 2n+1,\ n \text{ even} \\ 1, & \text{if } k = 2n+1,\ n \text{ odd} \end{cases}$$

$$\sin x = -(x-\pi) + \frac{1}{3!}(x-\pi)^3 - \frac{1}{5!}(x-\pi)^5 + \frac{1}{7!}(x-\pi)^7 - \cdots + (-1)^{n+1}\frac{1}{(2n+1)!}(x-\pi)^{2n+1} + \cdots$$

41. Diverges, because it is –5 times the harmonic series: $\sum_{n=1}^{\infty} \frac{-5}{n} = -5\sum_{n=1}^{\infty}\frac{1}{n}$

42. Converges conditionally

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series $\left(p = \frac{1}{2}\right)$, so $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ does not converge absolutely. Use the Alternating Series Test to check for conditional convergence:

(1) $u_n = \frac{1}{\sqrt{n}} > 0$

(2) $n+1 > n \Rightarrow \sqrt{n+1} > \sqrt{n} \Rightarrow \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$, so the u_n are decreasing.

(3) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$

Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converge.

43. Converges absolutely by the Direct Comparison Test, since $0 \le \frac{\ln n}{n^3} < \frac{1}{n^2}$ for $n \ge 1$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-Test with $p = 2$.

44. Converges absolutely by the Ratio Test, since

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n+2}{(n+1)!} \cdot \frac{n!}{n+1} = \lim_{n\to\infty} \frac{n+2}{(n+1)^2} = 0.$$

45. Converges conditionally

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$ diverges by the Limit Comparison Test. $\left(\text{Let } a_n = \frac{1}{\ln(n+1)}\right.$ and $b_n = \frac{1}{n}$.

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p-Test, and

$$c = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\ln(n+1)} = \lim_{n\to\infty} \frac{1}{1/(n+1)} = \lim_{n\to\infty} (n+1) = \infty.)$$

Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$ does not converge absolutely.
Use the Alternating Series Test to check for conditional convergence:

(1) $u_n = \frac{1}{\ln(n+1)} > 0$ Clear.

(2) $n+1 > n \Rightarrow \ln(n+1) > \ln n$
$\Rightarrow \frac{1}{\ln(n+1)} < \frac{1}{\ln n}$,
so the u_n are decreasing.

(3) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\ln n} = 0.$

Therefore, $\sum_{n=1}^{\infty} \frac{(-1)}{\ln n}$ converges.

46. Converges absolutely by the Integral Test, because

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} \left[-\frac{1}{\ln x}\right]_2^b = \frac{1}{\ln 2}.$$

47. Converges absolutely by the Ratio Test, because

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|-3|^{n+1}}{(n+1)!} \cdot \frac{n!}{|-3|^n} = \lim_{n\to\infty} \frac{3}{n+1} = 0.$$

48. Converges absolutely by the Direct Comparison Test, since $\frac{2^n 3^n}{n^n} \le \left(\frac{1}{2}\right)^n$ for $n \ge 12$ and $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is a convergent geometric series. Alternately, we may use the Ratio Test or the nth-Root Test (see Exercises 73 and 74 in Section 10.5).

49. Diverges by the nth-Term Test, since

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n^2+1}{2n^2+n-1} = \frac{1}{2} \ne 0.$$

50. Converges absolutely by the Direct Comparison Test, since

$$n(n+1)(n+2) = n^3 + 3n^2 + 2n > n^3$$
$$\Rightarrow \sqrt{n(n+1)(n+2)} > \sqrt{n^3}$$
$$\Rightarrow \frac{1}{\sqrt{n(n+1)(n+2)}} < \frac{1}{n^{3/2}},$$

and $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges by the p-Test.

51. Converges absolutely by the Limit Comparison Test:

Let $a_n = \frac{1}{n\sqrt{n^2-1}}$ and $b_n = \frac{1}{n^2}$. Then

$$c = \lim_{n\to\infty} \frac{a_n}{b_n}$$
$$= \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}}$$
$$= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}}$$
$$= 1$$

Since $0 < c < \infty$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-Test, $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2-1}}$ converges.

52. Diverges by the nth-Term Test, since $\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n} = \frac{1}{e} \neq 0.$

53. This is a telescoping series.

$$\sum_{n=3}^{\infty} \frac{1}{(2n-3)(2n-1)} = \sum_{n=3}^{\infty} \left(\frac{1}{2(2n-3)} - \frac{1}{2(2n-1)}\right)$$
$$s_1 = \frac{1}{2(2\cdot 3-3)} - \frac{1}{2(2\cdot 3-1)} = \frac{1}{6} - \frac{1}{10}$$
$$s_2 = \left(\frac{1}{6} - \frac{1}{10}\right) + \left(\frac{1}{10} - \frac{1}{14}\right) = \frac{1}{6} - \frac{1}{14}$$
$$s_3 = \left(\frac{1}{6} - \frac{1}{10}\right) + \left(\frac{1}{10} - \frac{1}{14}\right) + \left(\frac{1}{14} - \frac{1}{18}\right)$$
$$= \frac{1}{6} - \frac{1}{18}$$
$$s_n = \frac{1}{6} - \frac{1}{2(2(n+2)-1)} = \frac{1}{6} - \frac{1}{2(2n+3)}$$
$$S = \lim_{n\to\infty} s_n = \frac{1}{6}$$

54. This is a telescoping series.

$$\sum_{n=2}^{\infty} \frac{-2}{n(n+1)} = \sum_{n=2}^{\infty} \left(-\frac{2}{n} + \frac{2}{n+1}\right)$$
$$s_1 = -\frac{2}{2} + \frac{2}{3} = -1 + \frac{2}{3}$$
$$s_2 = \left(-1 + \frac{2}{3}\right) + \left(-\frac{2}{3} + \frac{2}{4}\right) = -1 + \frac{2}{4}$$
$$s_3 = \left(-1 + \frac{2}{3}\right) + \left(-\frac{2}{3} + \frac{2}{4}\right) + \left(-\frac{2}{4} + \frac{2}{5}\right)$$
$$= -1 + \frac{2}{5}$$
$$s_n = -1 + \frac{2}{n+2}$$
$$S = \lim_{n\to\infty} s_n = -1$$

55. **(a)** $P_3(x) = f(3) + f'(3)(x-3) + \frac{f''(3)}{2!}(x-3)^2 + \frac{f'''(3)}{3!}(x-3)^3$

$= 1 + 4(x-3) + 3(x-3)^2 + 2(x-3)^3$

$f(3.2) \approx P_3(3.2) = 1.936$

(b) Since the Taylor series for f' can be obtained by term-by-term differentiation of the Taylor Series for f, the second order Taylor polynomial for f' at $x = 3$ is $4 + 6(x-3) + 6(x-3)^2$. Evaluated at $x = 2.7$, $f'(2.7) \approx 2.74$.

(c) It underestimates the values, since $f''(3) = 6$, which means the graph of f is concave up near $x = 3$.

56. **(a)** Since the constant term is $f(4)$, $f(4) = 7$. Since $-2 = \frac{f'''(4)}{3!}$, $f'''(4) = -12$.

(b) Note that $P_4'(x) = -3 + 10(x-4) - 6(x-4)^2 + 24(x-4)^3$. The second order polynomial for f' at $x = 4$ is given by the first three terms of this expression, namely $-3 + 10(x-4) - 6(x-4)^2$. Evaluating at $x = 4.3$, $f'(4.3) \approx -0.54$.

(c) The fourth order Taylor polynomial for $g(x)$ at $x = 4$ is

$$\int_4^x [7 - 3(t-4) + 5(t-4)^2 - 2(t-4)^3]\, dx = \left[7t - \frac{3}{2}(t-4)^2 + \frac{5}{3}(t-4)^3 - \frac{1}{2}(t-4)^4\right]_4^x$$

$$= 7(x-4) - \frac{3}{2}(x-4)^2 + \frac{5}{3}(x-4)^3 - \frac{1}{2}(x-4)^4$$

(d) No; one would need the entire Taylor series for $f(x)$, and it would have to converge to $f(x)$ at $x = 3$.

57. **(a)** Use the Maclaurin series for $\sin x$ given at the end of Section 10.2.

$$5\sin\left(\frac{x}{2}\right) = 5\left[\frac{x}{2} - \frac{(x/2)^3}{3!} + \frac{(x/2)^5}{5!} - \cdots + (-1)^n \frac{(x/2)^{2n+1}}{(2n+1)!} + \cdots\right]$$

$$= \frac{5x}{2} - \frac{5x^3}{48} + \frac{x^5}{768} - \cdots + (-1)^n \frac{5}{(2n+1)!}\left(\frac{x}{2}\right)^{2n+1} + \cdots$$

(b) The series converges for all real numbers, according to the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{5}{(2n+3)!}\left|\frac{x}{2}\right|^{2n+3} \cdot \frac{(2n+1)!}{5}\left|\frac{2}{x}\right|^{2n+1}$$

$$= \lim_{n\to\infty} \frac{\left|\frac{x}{2}\right|^2}{(2n+3)(2n+2)}$$

$$= 0$$

(c) Note that the absolute value of $f^{(n)}(x)$ is bounded by $\frac{5}{2^n}$ for all x and all $n = 1, 2, 3, \cdots$.

We may use the Remainder Estimation Theorem with $M = 5$ and $r = \frac{1}{2}$.

So if $-2 < x < 2$, the truncation error using P_n is bounded by $\frac{5}{2^{n+1}} \cdot \frac{2^{n+1}}{(n+1)!} = \frac{5}{(n+1)!}$.

To make this less than 0.1 requires $n \geq 4$. So, two terms (up through degree 4) are needed.

58. (a) Substitute $2x$ for x in the Maclaurin series for $\frac{1}{1-x}$ given at the end of Section 10.2.

$$\begin{aligned}&\frac{1}{1-2x}\\&=1+2x+(2x)^2+(2x)^3+\cdots+(2x)^n+\cdots\\&=1+2x+4x^2+8x^3+\cdots+(2x)^n+\cdots\end{aligned}$$

(b) $\left(-\frac{1}{2}, \frac{1}{2}\right)$. The series is geometric with $r = 2x$, so it converges for $|2x| < 1$. (You could also use the Ratio Test, but you would need to verify divergence at the endpoints)

(c) $f\left(-\frac{1}{4}\right)=\frac{2}{3}$, so one percent is approximately 0.0067. It takes 7 terms (up through degree 6). This can be found by trial and error. Also, for $x=-\frac{1}{4}$, the series is the alternating series $\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n$. If you use the Alternating Series Estimation Theorem, it shows that 8 terms (up through degree 7) are sufficient since $\left|-\frac{1}{2}\right|^8 < 0.0067.$

It is also a geometric series, and you could use the remainder formula for a geometric series to determine the number of terms needed. (See Example 2 in Section 10.3.)

59. (a)
$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{|x|^{n+1}(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{|x|^n n^n}\\&=\lim_{n\to\infty}\frac{|x|(n+1)^{n+1}}{(n+1)n^n}\\&=|x|\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n\\&=|x|e\end{aligned}$$

The series converges for $|x|e<1$, or $|x|<\frac{1}{e}$, so the radius of convergence is $\frac{1}{e}$.

(b)
$$\begin{aligned}f\left(-\frac{1}{3}\right)&\approx-\frac{1}{3}\cdot\frac{1}{1}+\left(-\frac{1}{3}\right)^2\cdot\frac{2^2}{2!}+\left(-\frac{1}{3}\right)^3\cdot\frac{3^3}{3!}\\&=-\frac{1}{3}+\frac{2}{9}-\frac{1}{6}\\&=-\frac{5}{18}\approx-0.278\end{aligned}$$

(c) By the Alternating Series Estimation Theorem the error is no more than the magnitude of the next term, which is $\left|\left(-\frac{1}{3}\right)^4\cdot\frac{4^4}{4!}\right|=\frac{32}{243}\approx 0.132.$

60. **(a)** $f(3) = (x-2)^{-1}\big|_{x=3} = 1$

$f'(3) = -(x-2)^{-2}\big|_{x=3} = -1$

$f''(3) = 2(x-2)^{-3}\big|_{x=3} = 2, \text{ so } \dfrac{f''(3)}{2!} = 1$

$f'''(3) = -6(x-2)^{-4}\big|_{x=3} = -6, \text{ so } \dfrac{f'''(3)}{3!} = -1$

$f^{(n)}(3) = (-1)^n n!, \text{ so } \dfrac{f^{(n)}(3)}{n!} = (-1)^n$

$f(x) = 1-(x-3)+(x-3)^2-(x-3)^3+\cdots+(-1)^n(x-3)^n+\cdots$

(b) Integrate term by term.

$$\begin{aligned}\ln|x-2| &= \int_3^x \frac{1}{t-2}\,dt\\ &= \int_3^x \left(1-(t-3)+(t-3)^2-(t-3)^3+\cdots+(-1)^n(t-3)^n+\cdots\right)dt\\ &= \left[t-\frac{1}{2}(t-3)^2+\frac{1}{3}(t-3)^3-\frac{1}{4}(t-3)^4+\cdots+(-1)^n\frac{(t-3)^{n+1}}{n+1}+\cdots\right]_3^x\\ &= (x-3)-\frac{(x-3)^2}{2}+\frac{(x-3)^3}{3}-\frac{(x-3)^4}{4}+\cdots+(-1)^n\frac{(x-3)^{n+1}}{n+1}+\cdots\end{aligned}$$

(c) Evaluate at $x = 3.5$. This is the alternating series $\frac{1}{2}-\frac{1}{2^2\cdot 2}+\frac{1}{2^3\cdot 3}-\cdots+(-1)^n\frac{1}{2^{n+1}(n+1)}+\cdots$ By the Alternating Series Estimation Theorem, since the size of the third term is $\frac{1}{24} < 0.05,$ the first two terms will suffice. The estimate for $\ln\left(\frac{3}{2}\right)$ is 0.375.

61. **(a)** Substitute $-2x^2$ for x in the Maclaurin series for e^x given at the end of Section 10.2.

$$\begin{aligned}e^{-2x^2} &= 1+(-2x^2)+\frac{(-2x^2)^2}{2!}+\frac{(-2x^2)^3}{3!}+\cdots+\frac{(-2x^2)^n}{n!}+\cdots\\ &= 1-2x^2+2x^4-\frac{4x^6}{3}+\cdots+(-1)^n\frac{2^n x^{2n}}{n!}+\cdots\end{aligned}$$

(b) Use the Ratio Test:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{2^{n+1}x^{2n+2}}{(n+1)!}\cdot\frac{n!}{2^n x^{2n}}\\ &= \lim_{n\to\infty}\frac{2x^2}{n+1}\\ &= 0\end{aligned}$$

The series converges for all real numbers, so the interval of convergence is $(-\infty, \infty)$.

(c) The difference between $f(x)$ and $g(x)$ is the truncation error. Since the series is an alternating series, the error is bounded by the mangnitude of the fifth term: $\frac{(2x^2)^4}{4!} = \frac{2x^8}{3}.$ Since $-0.6 \le x \le 0.6,$ this term is less than $\frac{2(0.6)^8}{3}$ which is less than 0.02.

62. **(a)** $f(x) = x^2\left(\frac{1}{1+x}\right)$

$$= x^2(1 - x + x^2 + \cdots + (-x)^n + \cdots)$$
$$= x^2 - x^3 + x^4 + \cdots + (-1)^n x^{n+2} + \cdots$$

(b) No; at $x = 1$, the series is $\sum_{n=0}^{\infty}(-1)^n$ and the partial sums form the sequence 1, 0, 1, 0, 1, 0, ..., which has no limit.

63. **(a)** Substituting t^2 for x in the Maclaurin series for $\sin x$ given at the end of Section 10.2,

$$\sin t^2 = t^2 - \frac{t^6}{3!} + \frac{t^{10}}{5!} - \cdots + (-1)^n \frac{t^{4n+2}}{(2n+1)!}.$$

Integrating term-by-term and observing that the constant term is 0,

$$\int_0^x \sin t^2\, dt = \frac{x^3}{3} - \frac{x^7}{7(3!)} + \frac{x^{11}}{11(5!)} - \cdots + (-1)^n \frac{x^{4n+3}}{(4n+3)(2n+1)!} + \cdots$$

(b) $\int_0^1 \sin x^2\, dx = \frac{1}{3} - \frac{1}{7(3!)} + \frac{1}{11(5!)} - \cdots + (-1)^n \frac{1}{(4n+3)(2n+1)!} + \cdots.$

Since the third term is $\frac{1}{11(5!)} = \frac{1}{1320} < 0.001$, it suffices to use the first two nonzero terms (through degree 7).

(c) $\text{NINT}(\sin x^2, x, 0, 1) \approx 0.31026830$

(d) $\frac{1}{3} - \frac{1}{7(3!)} + \frac{1}{11(5!)} - \frac{1}{15(7!)} = \frac{258{,}019}{831{,}600}$

$$\approx 0.31026816$$

This is within 1.5×10^{-7} of the answer in (c).

64. **(a)** Let $f(x) = x^2 e^x dx.$

$$\int_0^1 x^2 e^x\, dx = \int_0^1 f(x)\, dx$$
$$\approx \frac{h}{2}[f(0) + 2f(0.5) + f(1)]$$
$$= \frac{1}{4}\left[0 + 2\frac{e^{0.5}}{4} + e\right]$$
$$= \frac{e^{0.5}}{8} + \frac{e}{4}$$
$$\approx 0.88566$$

(b) $x^2e^x = x^2\left(1+x+\dfrac{x^2}{2!}+\cdots+\dfrac{x^n}{n!}+\cdots\right)$

$$= x^2+x^3+\frac{x^4}{2!}+\cdots+\frac{x^{n+2}}{n!}+\cdots$$

$$P_4(x)=x^2+x^3+\frac{x^4}{2}$$

$$\int_0^1 P_4(x)=\left[\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{10}\right]_0^1$$

$$=\frac{41}{60}$$

$$\approx 0.68333$$

(c) Since f is concave up, the trapezoids used to estimate the area lie above the curve, and the estimate is too large.

(d) Since all the derivatives are positive (and $x > 0$), the remainder, $R_n(x)$, must be positive. This means that $P_n(x)$ is smaller than $f(x)$.

(e) Let $u = x^2$ $\qquad dv = e^x dx$

$du = 2x\,dx$ $\qquad v = e^x$

$$\int x^2e^x dx = x^2e^x - \int 2x\,e^x\,dx$$

Let $u = 2x$ $\qquad dv = e^x dx$

$du = 2\,dx$ $\qquad v = e^x$

$$x^2e^x - \int 2xe^x\,dx = x^2e^x - \left[2xe^x - \int 2e^x\,dx\right]$$

$$= x^2e^x - 2xe^x + 2e^x + C$$

$$= (x^2-2x+2)e^x + C$$

$$\int_0^1 x^2e^x\,dx = (x^2-2x+2)e^x\Big]_0^1$$

$$= e-2 \approx 0.71828$$

65. (a) Because $[\$1000(1.08)^{-n}](1.08)^n = \$1000$ will be available after n years.

(b) Assume that the first payment goes to the charity at the end of the first year.
$1000(1.08)^{-1}+1000(1.08)^{-2}+1000(1.08)^{-3}+\cdots$

(c) This is a geometric series with sum equal to $\dfrac{\frac{1000}{1.08}}{1-\left(\frac{1}{1.08}\right)}=\dfrac{1000}{0.08}=12{,}500.$ This means that \$12,500 should be invested today in order to completely fund the perpetuity forever.

66. We again assume that the first payment occurs at the end of the year.

$$\begin{aligned}\text{Present value} &= 1000(1.06)^{-1} + 1000(1.06)^{-2} + 1000(1.06)^{-3} + \cdots \\ &= \frac{\frac{1000}{1.06}}{1-\left(\frac{1}{1.06}\right)} \\ &= \frac{1000}{1.06-1} \\ &\approx 16{,}666.67\end{aligned}$$

The present value is \$16,666.67.

67. **(a)**

Sequence of Tosses	Payoff (\$)	Probability	Term of Series
T	0	$\frac{1}{2}$	$0\left(\frac{1}{2}\right)$
HT	1	$\left(\frac{1}{2}\right)^2$	$1\left(\frac{1}{2}\right)^2$
HHT	2	$\left(\frac{1}{2}\right)^3$	$2\left(\frac{1}{2}\right)^3$
HHHT	3	$\left(\frac{1}{2}\right)^4$	$3\left(\frac{1}{2}\right)^4$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$$\text{Expected payoff } = 0\left(\frac{1}{2}\right) + 1\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^4 + \cdots + n\left(\frac{1}{2}\right)^{n+1} + \cdots$$

(b) $$\begin{aligned}\frac{1}{(1-x)^2} &= \frac{d}{dx}\left(1 + x + x^2 + x^3 + \cdots + x^n + \cdots\right) \\ &= 1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots\end{aligned}$$

(c) $$\begin{aligned}\frac{x^2}{(1-x)^2} &= x^2(1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots) \\ &= x^2 + 2x^3 + 3x^4 + \cdots + nx^{n+1} + \cdots\end{aligned}$$

(d) If $x = \frac{1}{2}$, the formula in part (c) matches the nonzero terms of the series in part (a). Since $\frac{\left(\frac{1}{2}\right)^2}{\left[1-\left(\frac{1}{2}\right)\right]^2} = 1$, the expected payoff is \$1.

68. **(a)** The area of an equilateral triangle whose sides have length s is $\frac{1}{2}(s)\left(\frac{\sqrt{3}s}{2}\right)=\frac{s^2\sqrt{3}}{4}$. The sequence of areas removed from the original triangle is

$$\frac{b^2\sqrt{3}}{4}+3\left(\frac{b}{2}\right)^2\frac{\sqrt{3}}{4}+9\left(\frac{b}{4}\right)^2\frac{\sqrt{3}}{4}+\cdots+3^n\left(\frac{b}{2^n}\right)^2\frac{\sqrt{3}}{4}+\cdots \text{ or}$$

$$\frac{b^2\sqrt{3}}{4}+\frac{3b^2\sqrt{3}}{4^2}+\frac{3^2b^2\sqrt{3}}{4^3}+\cdots+\frac{3^nb^2\sqrt{3}}{4^{n+1}}+\cdots.$$

(b) This is a geometric series with initial term $a=\frac{b^2\sqrt{3}}{4}$ and common ratio $r=\frac{3}{4}$, so the sum is $\frac{\frac{b^2\sqrt{3}}{4}}{1-\frac{3}{4}}=b^2\sqrt{3}$, which is the same as the area of the original triangle.

(c) No. For example, let $b=\frac{1}{2}$ and set the base of the original triangle along the x-axis from (0, 0) to (1, 0). The points removed from the base are all of the form $\left(\frac{k}{2^n}, 0\right)$, so points of the form $(x, 0)$ with x irrational (among others) still remain. The same sort of thing happens along the other two sides of the original triangle, and, in fact, along the sides of any of the smaller remaining triangles. While there are infinitely many points remaining throughout the original triangle, they paradoxically take up zero area.

69. $\frac{1}{1-x}=1+x+x^2+x^3+\cdots+x^n+\cdots$

Differentiate both sides.

$$\frac{1}{(1-x)^2}=1+2x+3x^2+4x^3+5x^4+\cdots+nx^{n-1}+\cdots$$

Substitute $x=\frac{1}{2}$ to get the desired result.

70. **(a)** Note that $\sum_{n=1}^{\infty}x^{n+1}$ is a geometric series with first term $a=x^2$ and common ratio $r=x$, which explains the identity $\sum_{n=1}^{\infty}x^{n+1}=\frac{x^2}{1-x}$ (for $|x|<1$).

Differentiate.

$$\sum_{n=1}^{\infty}(n+1)x^n=\frac{(1-x)(2x)-(x^2)(-1)}{(1-x)^2}$$
$$=\frac{2x-x^2}{(1-x)^2}$$

Differentiate again,

$$\sum_{n=1}^{\infty} n(n+1)x^{n-1}$$
$$= \frac{(1-x)^2(2-2x)-(2x-x^2)[2(1-x)(-1)]}{(1-x)^4}$$
$$= \frac{(1-x)^2(2-2x)+2(1-x)(2x-x^2)}{(1-x)^4}$$
$$= \frac{(1-x)[(1-x)(2-2x)+2(2x-x^2)]}{(1-x)^4}$$
$$= \frac{2}{(1-x)^3}$$

Multiply by x.

$$\sum_{n=1}^{\infty} n(n+1)x^n = \frac{2x}{(1-x)^3}$$

Replace x by $\frac{1}{x}$.

$$\sum_{n=1}^{\infty} \frac{n(n+1)}{x^n} = \frac{\frac{2}{x}}{\left(1-\frac{1}{x}\right)^3} = \frac{2x^2}{(x-1)^3},\ |x| > 1$$

(b) Use a grapher to solve the equation

$$x = \frac{2x^2}{(x-1)^3}.$$

The grapher calculates that $x \approx 2.769$ is the solution of the equation that is greater than 1.

71. (a) Computing the coefficients,

$$f(1) = \frac{1}{2}$$

$f'(x) = -(x+1)^{-2}$, so $f'(1) = -\frac{1}{4}$

$f''(x) = 2(x+1)^{-3}$, so $\frac{f''(1)}{2!} = \frac{1}{8}$

$f'''(x) = -6(x+1)^{-4}$, so $\frac{f'''(1)}{3!} = -\frac{1}{16}$

In general, $f^{(n)}(x) = (-1)^n n!(x+1)^{-n-1}$,

so $\frac{f^{(n)}(1)}{n!} = \frac{(-1)^n}{2^{n+1}}$.

$f(x)$

$$= \frac{1}{2} - \frac{x-1}{4} + \frac{(x-1)^2}{8} - \cdots + (-1)^n \frac{(x-1)^n}{2^{n+1}} + \cdots$$

(b) Ratio test for absolute convergence:

$$\lim_{n\to\infty} \frac{|x-1|^{n+1}}{2^{n+2}} \bullet \frac{2^{n+1}}{|x-1|^n} = \frac{|x-1|}{2}$$

$\frac{|x-1|}{2} < 1 \Rightarrow -1 < x < 3$. The series converges absolutely on (–1, 3).

At $x = -1$, the series is $\sum_{n=0}^{\infty} \frac{1}{2}$, which diverges by the nth-term test.

At $x = 3$, the series is $\sum_{n=0}^{\infty} (-1)^n \frac{1}{2}$, which diverges by the nth-term test.

The interval of convergence is (–1, 3).

(c) $P_3(x) = \frac{1}{2} - \frac{x-1}{4} + \frac{(x-1)^2}{8} - \frac{(x-1)^3}{16}$

$P_3(0.5)$

$$= \frac{1}{2} - \frac{0.5-1}{4} + \frac{(0.5-1)^2}{8} - \frac{(0.5-1)^3}{16}$$
$$= 0.6640625$$

72. (a) Ratio test for absolute convergence:

$$\lim_{n\to\infty} \frac{(n+1)|x|^{n+1}}{2^{n+1}} \cdot \frac{2^n}{n|x|^n}$$
$$= \lim_{n\to\infty} \frac{n+1}{n} \frac{|x|}{2}$$
$$= \frac{|x|}{2}$$

$\frac{|x|}{2} < 1 \Rightarrow -2 < x < 2$

The series converges absolutely on (–2, 2).

The series diverges at both endpoints by the nth-term test, since

$\lim_{n\to\infty} n \neq 0$ and $\lim_{n\to\infty} (-1)^n n \neq 0$.

The interval of convergence is (–2, 2).

(b) The series converges at –1 and forms an alternating series:

$-\frac{1}{2} + \frac{2}{4} - \frac{3}{8} + \frac{4}{16} + \cdots (-1)^n \frac{n}{2^n} + \cdots$. The nth-term of this series decreases in absolute value to 0, so the truncation error after 9 terms is less than the absolute value of the 10^{th} term. Thus,

error $< \frac{10}{2^{10}} < 0.01$.

73. (a) $P_1(x) = -1 + 2x$

(b) $P_2(x) = -1 + 2x - \frac{3}{2}x^2$

(c) $P_3(x) = -1 + 2x - \frac{3}{2}x^2 + \frac{2}{3}x^3$

(d) $P_3(0.7) = -1 + 2(0.7) - \frac{3}{2}(0.7)^2 + \frac{2}{3}(0.7)^3$
≈ -0.1063